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# Mass transport in layered fluid systems

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The method of matched asymptotic expansions is employed to calculate the mass transport velocity due to small amplitude oscillatory waves propagating in conditions of density and viscosity discontinuities. For progressive waves in a two-layer system, it is found that the velocity at the interface is in the direction of wave propagation; when the uppermost surface is free, the velocity there is in the direction opposite to that at the interface. If the difference in the densities is small, the calculated transport velocity associated with an internal wave can be of more importance than that associated with the surface wave as obtained from the work of Longuet-Higgins (1953).

## 1. Introduction

The calculation of the principal contribution to the Lagrangian particle drift or mass transport velocity in a slightly viscous, incompressible, homogeneous fluid was carried out some years ago in a detailed study by Longuet-Higgins (1953). In this work consideration was given to both progressive and standing waves of small (but finite) amplitude propagating two dimensionally. An important feature of the solution for progressive waves was found to be a strong forwards velocity near the bottom; the profile of the transport velocity was dependent on the wavelength whilst the direction just below the free surface could be opposite to that of wave propagation.

In the present work we determine the principal contribution to the mass transport in fluid systems consisting of layers of homogeneous fluids. In particular, the case of two layers is studied but the uppermost surface may be a plane rigid boundary or may be free. The method of analysis involves a double expansion in powers of two small parameters associated with the wave amplitude and fluid viscosities together with notions of boundary-layer theory. Essentially, the parameters are a measure of the ratio of wave amplitude to wavelength and of the inverse wave Reynolds number. An expansion in powers of the former parameter only was carried out many years ago for a single viscous fluid by Harrison (1909).

The case of progressive waves is here analyzed explicitly and it is found that the mass transport velocity can formally be an order of magnitude larger than that obtained by Longuet-Higgins (1953) for a single homogeneous fluid. Under conditions of zero total horizontal flow in each fluid, the drift at the interface is shown to be in the direction of wave propagation whilst that at a free upper surface is in the opposite direction. In addition, the profile of the transport velocity in either fluid is found to be independent of wavelength.

## 2. Formulation

We refer the equations of motion to Cartesian co-ordinates $(x, z)$ fixed in the equilibrium level of the surface of discontinuity $z = 0$ separating two homogeneous incompressible fluids, the $z$ axis being directed vertically upwards. The region of flow is assumed to be bounded below by the fixed horizontal plane $z = -h^{(1)}$ whilst the upper boundary is either assumed to be the fixed plane $z = h^{(2)}$ or is assumed to be free. We suppose that the density of the lower fluid is greater than that of the upper fluid. The equations of laminar motion for either fluid are

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\,\mathbf{q} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{q}, \tag{2.1}$$

where $\mathbf{q} = (u, w)$, $p$, $\rho$ and $\nu$ denote fluid velocity, change in pressure from the equilibrium state, density and kinematic viscosity respectively. Defining a stream function $\psi$ such that

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x}, \tag{2.2}$$

it can readily be shown that

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial z}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial z}\right)\nabla^2\psi = \nu\nabla^4\psi. \tag{2.3}$$

We now propose to consider particular solutions of (2.3) representative of progressive waves in the case when the fluid viscosities are small in some sense. It will thus be possible to analyze the flow by appealing to well-known methods of boundary-layer theory as has been done, for example, in the works of Johns (1968) and Dore (1969) with reference to waves of infinitesimal amplitude. Let $\sigma$ be the real angular frequency of oscillation and $k$ be the complex wave-number allowing for spatial attenuation of the waves. Then equation (2.3) may be non-dimensionalized according to the scheme

$$\hat{\mathbf{r}} = k_0\mathbf{r}, \quad \hat{t} = \sigma t, \quad \hat{\psi} = (k_0^2/\sigma)\psi, \quad \hat{\epsilon} = (\nu k_0^2/\sigma)^{\frac{1}{2}}. \tag{2.4}$$

The quantity $k_0$ is the wave-number according to the inviscid theory of waves of infinitesimal amplitude, whilst $\hat{\epsilon}$ is the inverse wave Reynolds number. In the expansion scheme to follow, we shall assume $\hat{\epsilon} \ll 1$ but, if the waves are long relative to a fluid depth $h^{(r)}$, it will be further necessary that $\hat{\epsilon} \ll k_0 h^{(r)}$, $(r = 1, 2)$. Henceforth, we omit the symbol $(\hat{\ })$, so that the non-dimensional form of equation (2.3) is

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial z}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial z}\right)\nabla^2\psi = \epsilon^2\nabla^4\psi. \tag{2.5}$$

The kinematic viscosity $\nu$, and hence $\epsilon$, may be different in the two fluids.

The analysis will first be carried out by employing the methods of matched asymptotic expansions, but a less formal derivation of the principal result will also be indicated in a later section.

## 3. Consideration of the boundary layers

Within the viscous layers adjacent to the oscillating interface, $z = \zeta(x,t)$, we write

$$\psi(x,Z,t;\epsilon) = \alpha\psi_1(Z;\epsilon)\exp\{i(kx-\sigma t)\} + \alpha^2\psi_2(x,Z,t;\epsilon)+\ldots, \tag{3.1}$$

$$\psi_1 = \psi_{10}(Z)+\epsilon\psi_{11}(Z)+\epsilon^2\psi_{12}(Z)+\ldots, \tag{3.2}$$

$$\psi_2 = \epsilon^{-1}\psi_{2,-1}(x,Z,t)+\psi_{20}(x,Z,t)+\epsilon\psi_{21}(x,Z,t)+\ldots, \tag{3.3}$$

$$k = k_0+\epsilon k_1+\ldots, \tag{3.4}$$

where
$$z = \epsilon Z, \tag{3.5}$$

$\sigma = 1 = k_0$ here and $\alpha$ is an ordering parameter (based on the ratio of wave amplitude to wavelength). Only the real parts of complex quantities on the right-hand side of (3.1) are physically significant. The presence of the term $\psi_{2,-1}$ may seem surprising but it does not appear to be immediately obvious that it can be omitted.

Substitution of the above equations into (2.5) yields the following equations for $\psi_{10}$, $\psi_{11}$, $\psi_{12}$:

$$\mathscr{L}\psi_{10} \equiv \psi_{10}^{\mathrm{iv}}+i\psi_{10}'' = 0, \tag{3.6}$$

$$\mathscr{L}\psi_{11} = 0, \quad \mathscr{L}\psi_{12} = i\psi_{10}, \tag{3.7}$$

where dashes denote differentiation with respect to $Z$ and use has been made in (3.7) of the result that $\psi_{10}$ = constant by linear theory ($\psi_{10}' \neq 0$ would imply horizontal particle velocities of order $\alpha\epsilon^{-1}$). We shall later require the solutions

$$\psi_{11}^{(1)} = a_{11}^{(1)}\exp\{-nZ^{(1)}\}+c_{11}^{(1)}+d_{11}^{(1)}Z^{(1)}, \tag{3.8}$$

$$\psi_{11}^{(2)} = a_{11}^{(2)}\exp\{nZ^{(2)}\}+c_{11}^{(2)}+d_{11}^{(2)}Z^{(2)}, \tag{3.9}$$

where $n = (-1+i)/\sqrt{2}$ and superscripts (1) and (2) denote quantities in the lower and upper fluid respectively. We further note that $\psi_{11}''$ and $\psi_{12}'''$ are exponentially small as we approach the outer edges of the interfacial layers. The quantities $\psi_{2,-1}$, $\psi_{20}$, ... (which are of second order in the amplitude parameter) contain parts which are independent of the time and which have the decay factor $\exp\{i(k-k^*)x\}$, the asterisk denoting a complex conjugate. Omitting this factor for the moment and indicating the remaining steady part of $\psi_{2,-1}$, $\psi_{20}$, ... by a bar, we obtain

$$\overline{\psi}_{2,-1}^{\mathrm{iv}} = 0, \tag{3.10}$$

$$\overline{\psi}_{20}^{\mathrm{iv}} = \tfrac{1}{2}i\psi_{10}^{*}\psi_{11}''', \tag{3.11}$$

$$\overline{\psi}_{21}^{\mathrm{iv}} = \tfrac{1}{2}i[(k_0\psi_{12}'''+k_1^*\psi_{11}''')\psi_{10}^*+k_0(\psi_{11}^*\psi_{11}'')']. \tag{3.12}$$

In deriving these equations, we have made use of the result

$$\overline{\mathrm{Re}(P)\,\mathrm{Re}(Q)} = \tfrac{1}{2}\mathrm{Re}(PQ^*) = \tfrac{1}{2}\mathrm{Re}(P^*Q), \tag{3.13}$$

$P, Q$ being any two complex quantities with a time period $2\pi/\sigma$.

The interfacial boundary conditions of interest are the kinematical condition together with continuity of velocity and tangential stress across the interface $z = \zeta(x,t)$. These conditions will be satisfied (to the required order) by means of a direct expansion about the equilibrium interfacial level $z = 0$. As a consequence, the restriction

$$\alpha \ll \epsilon \tag{3.14}$$

on wave amplitude must be imposed, for the reasons stated by Longuet-Higgins (1953). If we write

$$\zeta = \alpha(\zeta_{10} + \epsilon\zeta_{11} + \ldots)\exp\{i(kx - \sigma t)\} + \alpha^2\zeta_2(x,t) + \ldots, \tag{3.15}$$

the kinematical condition

$$w = \frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} \quad \text{on} \quad z = \zeta$$

yields

$$\overline{\psi}_{2,-1} = 0, \tag{3.16}$$

on $Z = 0$. To ensure that the horizontal velocity $u$ be continuous across the interface, we shall also require the quantities

$$\epsilon^{-2}\overline{\psi}'_{2,-1}, \quad \epsilon^{-1}(\overline{\psi}'_{20} + \tfrac{1}{2}\zeta^*_{10}\psi''_{11}), \tag{3.17}$$

to be continuous on $Z = 0$. Lastly, to the second order in amplitude, the tangential stress condition may be written

$$\mu\left[\left(\frac{\partial^2\psi}{\partial z^2} - \frac{\partial^2\psi}{\partial x^2}\right) - 4\frac{\partial\zeta}{\partial x}\frac{\partial^2\psi}{\partial x\partial z}\right] \quad \text{continuous on} \quad z = \zeta \tag{3.18}$$

(see, for example, Longuet-Higgins 1960), $\mu = \rho\nu$ being the fluid viscosity. Expanding this condition about the equilibrium level, $z = 0$, of the interface then demands that

$$\mu\epsilon^{-3}\overline{\psi}''_{2,-1}, \quad \mu\epsilon^{-2}(\overline{\psi}''_{20} + \tfrac{1}{2}\zeta^*_{10}\psi'''_{11}), \quad \mu\epsilon^{-1}[\overline{\psi}''_{21} + \tfrac{1}{2}(\zeta^*_{10}\psi'''_{12} + \zeta^*_{11}\psi'''_{11})], \tag{3.19}$$

be continuous on $Z = 0$. The quantity $\zeta_{11}$ can effectively be regarded as zero for the two-layer fluid system considered here (if the interfacial amplitude is taken to be prescribed) but would definitely be needed for three or more layers.

In the flow outside the boundary layers the horizontal Eulerian mean velocity will be seen to be of order $\alpha^2\epsilon^{-1}$. In consequence, asymptotic matching will require that

$$\overline{\psi}'_{2,-1}; \quad \overline{\psi}''_{2,-1}, \overline{\psi}''_{20}; \quad \overline{\psi}'''_{2,-1}, \overline{\psi}'''_{20}, \overline{\psi}'''_{21}; \tag{3.20}$$

be negligibly small as we approach the outer edges of the interfacial layers. Thus, integrations of (3.10), (3.11) and (3.12) together with use of (3.7) give

$$\left.\begin{aligned} \overline{\psi}_{2,-1} &= 0, \\ \overline{\psi}'_{20} &= -\tfrac{1}{2}\psi^*_{10}\psi''_{11} + b_{20}, \\ \overline{\psi}''_{21} &= -\tfrac{1}{2}\psi^*_{10}\psi'''_{12} + \tfrac{1}{2}i(k^*_1\psi^*_{10} + k_0\psi^*_{11})\psi'_{11} + c_{21}, \end{aligned}\right\} \tag{3.21}$$

where we have satisfied the requirements (3.16) and (3.20). (After the first integration of (3.12), the imaginary quantity $\tfrac{1}{2}ik_0\psi^{*\prime}_{11}\psi'_{11}$ can be added to the right-hand side to facilitate the next integration, since only the real part of $\overline{\psi}_{21}$ is of physical interest.)

In the work of Longuet-Higgins (1953) for a single layer of fluid, application of the tangential stress condition enabled an explicit boundary condition to be determined for the derivative of the (lowest order) horizontal mass transport velocity in the interior of the fluid. The procedure is necessarily different in the present work and the constants $b_{20}$, $c_{21}$ in (3.21) can only be determined by a consideration of the Eulerian mean velocity over the whole region of the fluid. Thus, in the lower boundary layer adjacent to the rigid boundary $z = -h^{(1)}$, it is known from the analysis of Longuet-Higgins (1953) and others that the lowest order horizontal Eulerian mean velocity is $O(\alpha^2)$. The same is true in the uppermost boundary layer when it is adjacent to the rigid boundary $z = h^{(2)}$. If the top surface is free, however, with equation $z = h^{(2)} + \zeta^{(2)}$, the equations analogous to (3.21) for the viscous surface layer are

$$\overline{\psi}_{2,-1} = 0, \quad \overline{\psi}'_{20} = \beta_{20}, \quad \overline{\psi}''_{21} = -\tfrac{1}{2}\psi^*_{10}\psi'''_{12}, \tag{3.22}$$

the dashes denoting differentiation with respect to the variable $(z-h^{(2)})/\epsilon^{(2)}$, since $\psi'_{11}$ = constant for such a layer with clean free surface (that is, the lowest order horizontal velocity, $O(\alpha)$, is constant through the layer). These results will now enable us to readily calculate the lowest order drift velocity, $O(\alpha^2\epsilon^{-1})$, in the interior of the fluid system.

## 4. Consideration of the outer flow

In the flow outside the above mentioned boundary layers we write

$$\Psi(x,z,t;\epsilon) = \alpha\Psi_1(z;\epsilon)\exp\{i(kx-\sigma t)\} + \alpha^2\Psi_2(x,z,t;\epsilon) + \ldots, \tag{4.1}$$

$$\Psi_1 = \Psi_{10}(z) + \epsilon\Psi_{11}(z) + \ldots, \tag{4.2}$$

$$\Psi_2 = \epsilon^{-1}\Psi_{2,-1}(x,z,t) + \Psi_{20}(x,z,t) + \ldots, \tag{4.3}$$

analogous to equations (3.1), (3.2) and (3.3). Then, when $\alpha \ll \epsilon$, we have

$$\nabla^4\overline{\Psi}_2 = 0 \tag{4.4}$$

as in the conduction solution of Longuet-Higgins (1953). In order to match with the solutions in the interfacial layers, the steady part of $\Psi_{2,-1}$, $\Psi_{20}$, ... must possess the factor $\exp\{i(k-k^*)x\}$. Leaving this aside for the moment we have

$$\frac{\partial^4\overline{\Psi}_{2,-1}}{\partial z^4} = 0, \quad \overline{\Psi}_{2,-1} = A_{2,-1} + B_{2,-1}z + C_{2,-1}z^2 + D_{2,-1}z^3. \tag{4.5}$$

The boundary conditions on $\overline{\Psi}^{(r)}_{2,-1}$ are that

$$\overline{\Psi}^{(1)}_{2,-1} = 0 = \frac{\partial\overline{\Psi}^{(1)}_{2,-1}}{\partial z} \quad \text{on} \quad z = -h^{(1)}, \tag{4.6}$$

$$\overline{\Psi}^{(2)}_{2,-1} = 0 = \frac{\partial\overline{\Psi}^{(2)}_{2,-1}}{\partial z} \quad \text{on} \quad z = h^{(2)}, \tag{4.7a}$$

for a fluid system contained between rigid horizontal boundaries, whilst if the uppermost surface is free, (3.22) shows that the conditions (4.7 $a$) must be replaced by

$$\Psi^{(2)}_{2,-1} = 0 = \frac{\partial^2 \Psi^{(2)}_{2,-1}}{\partial z^2} \quad \text{on} \quad z = h^{(2)}. \tag{4.7b}$$

The constant $\beta_{20}$ in equation (3.22) can be determined by asymptotic matching only when the complete solution for $\Psi^{(2)}_{2,-1}$ has been obtained. On the equilibrium level $z = 0$ of the interface, matching techniques show that

$$\overline{\Psi}^{(r)}_{2,-1} = 0, \tag{4.8}$$

$$\frac{\partial \overline{\Psi}^{(r)}_{2,-1}}{\partial z} = \overline{\psi}^{(r)\prime}_{20}(Z^{(r)}_{\infty}), \tag{4.9}$$

$$\frac{\partial^2 \overline{\Psi}^{(r)}_{2,-1}}{\partial z^2} = \overline{\psi}^{(r)\prime\prime}_{21}(Z^{(r)}_{\infty}), \tag{4.10}$$

($r = 1, 2$), where the outer edges of the interfacial layers are here characterized by $Z = Z_\infty$. Thus, we have here 10 conditions which, together with the requirements (3.17) and (3.19) on $\overline{\psi}'_{20}$ and $\overline{\psi}''_{21}$ respectively, yield 12 equations for the 12 unknowns $A^{(r)}_{2,-1}$, $B^{(r)}_{2,-1}$, $C^{(r)}_{2,-1}$, $D^{(r)}_{2,-1}$, $b^{(r)}_{20}$, $c^{(r)}_{21}$.

### (*a*) *Fluid bounded by horizontal planes*

For this case boundary conditions (4.7 $a$) apply and, with a slight adjustment in the notation, we easily find

$$\Psi^{(1)}_{2,-1} = D^{(1)} z\left(1 + \frac{z}{h^{(1)}}\right)^2, \quad \Psi^{(2)}_{2,-1} = D^{(2)} z\left(1 - \frac{z}{h^{(2)}}\right)^2, \tag{4.11}$$

$$\frac{\partial \overline{\Psi}^{(1)}_{2,-1}}{\partial z} = D^{(1)}\left(1 + \frac{z}{h^{(1)}}\right)\left(1 + \frac{3z}{h^{(1)}}\right), \quad \frac{\partial \overline{\Psi}^{(2)}_{2,-1}}{\partial z} = D^{(2)}\left(1 - \frac{z}{h^{(2)}}\right)\left(1 - \frac{3z}{h^{(2)}}\right), \tag{4.12}$$

where
$$\frac{D^{(r)}}{\epsilon^{(r)}} = \frac{1}{8}\left[\frac{\mu^{(2)}}{\epsilon^{(2)}}\psi^{*(2)\prime\prime}_{11}(0)\psi^{(2)\prime}_{11}(Z^{(2)}_{\infty}) - \frac{\mu^{(1)}}{\epsilon^{(1)}}\psi^{*(1)\prime\prime}_{11}(0)\psi^{(1)\prime}_{11}(Z^{(1)}_{\infty})\right] \Big/ \left(\frac{\mu^{(1)}}{h^{(1)}} + \frac{\mu^{(2)}}{h^{(2)}}\right). \tag{4.13}$$

Using the result, from the theory for waves of infinitesimal amplitude, that

$$a^{(r)}_{11} n^2 = \psi^{(r)\prime\prime}_{11}(0) = \frac{n}{\sqrt{(\rho^{(r)}\mu^{(r)})}} \frac{\sqrt{(\rho^{(1)}\mu^{(1)}\rho^{(2)}\mu^{(2)})}}{\sqrt{(\rho^{(1)}\mu^{(1)})} + \sqrt{(\rho^{(2)}\mu^{(2)})}}\Delta, \tag{4.14}$$

$$\Delta = \left[\frac{\partial \Psi^{(1)}_{10}}{\partial z} - \frac{\partial \Psi^{(2)}_{10}}{\partial z}\right]_{z=0}, \tag{4.15}$$

[see, for example, Dore 1969], we thus have

$$D^{(r)} = \theta\sqrt{(\nu^{(r)})}\sqrt{(\rho^{(1)}\mu^{(1)}\rho^{(2)}\mu^{(2)})}/(\sqrt{(\rho^{(1)}\mu^{(1)})} + \sqrt{(\rho^{(2)}\mu^{(2)})}), \tag{4.16}$$

$$\theta = \frac{\sqrt{2}}{16}(1+i)\Delta\Delta^* \Big/ \left(\frac{\mu^{(1)}}{h^{(1)}} + \frac{\mu^{(2)}}{h^{(2)}}\right), \tag{4.17}$$

where, of course, only the real part of $D^{(r)}$ is physically meaningful. We note here that

$$\mathrm{Re}(D) \geqslant 0. \tag{4.18}$$

(*b*) *Fluid with free upper surface*

In this case, we employ boundary conditions (4.7*b*), and find

$$\Psi^{(1)}_{2,-1} = D^{(1)}z\left(1+\frac{z}{h^{(1)}}\right)^2, \quad \Psi^{(2)}_{2,-1} = \tfrac{1}{2}D^{(2)}z\left(1-\frac{z}{h^{(2)}}\right)\left(1-\frac{2z}{h^{(2)}}\right), \tag{4.19}$$

$$\frac{\partial\Psi^{(1)}_{2,-1}}{\partial z} = D^{(1)}\left(1+\frac{z}{h^{(1)}}\right)\left(1+\frac{3z}{h^{(1)}}\right), \quad \frac{\partial\Psi^{(2)}_{2,-1}}{\partial z} = \tfrac{1}{2}D^{(2)}\left(2-\frac{6z}{h^{(2)}}+\frac{3z^2}{h^{(2)2}}\right), \tag{4.20}$$

$$D^{(r)} = \theta\sqrt{(\nu^{(r)})}\sqrt{(\rho^{(1)}\mu^{(1)}\rho^{(2)}\mu^{(2)})}/(\sqrt{(\rho^{(1)}\mu^{(1)})}+\sqrt{(\rho^{(2)}\mu^{(2)})}), \tag{4.21}$$

$$\theta = \frac{\sqrt{2}}{16}(1+i)\Delta\Delta^*\bigg/\left(\frac{\mu^{(1)}}{h^{(1)}}+\frac{3}{4}\frac{\mu^{(2)}}{h^{(2)}}\right). \tag{4.22}$$

The present analysis has therefore revealed a horizontal Eulerian mean velocity of order $\alpha^2\epsilon^{-1}$ which exists throughout the whole fluid system. It is clear, however, that this velocity vanishes when $\rho^{(2)} = 0$ or when $\rho^{(1)} = \rho^{(2)}$. The associated velocity in the interfacial layers is obtained from

$$\overline{\psi}'_{20} = D - \tfrac{1}{2}\psi^*_{10}\psi''_{11} \tag{4.23}$$

and thus varies rapidly through the layers. Observe also that the total horizontal Eulerian transport, represented in the lower fluid (say) by

$$\overline{\int_{-h^{(1)}}^{\zeta} u\,dz},$$

is necessarily zero to the lowest order, $O(\alpha^2\epsilon^{-1})$, in each layer on account of equations (4.6), (4.7) and (4.8), the latter of which is a consequence of the kinematic condition (3.16). This will be commented on further in the next section.

The lowest order steady vorticity in the interfacial layers is $O(\alpha^2\epsilon^{-2})$ and is exponentially small on approaching the outer edges; the next term in the vorticity expansion is $O(\alpha^2\epsilon^{-1})$ and tends to the values

(*a*) $4D^{(1)}/h^{(1)}$ and $-4D^{(2)}/h^{(2)}$,

(*b*) $4D^{(1)}/h^{(1)}$ and $-3D^{(2)}/h^{(2)}$,

at the outer edges of the lower and upper interfacial layers. This latter term is therefore responsible for the generation of vorticity, $O(\alpha^2\epsilon^{-1})$, in the outer flow through the medium of viscous diffusion. The vorticity so generated is thus dependent on $\nu$.

## 5. Mass transport

According to Longuet-Higgins (1953), the horizontal Lagrangian or mass transport velocity $U_l$ is given by

$$U_l = \overline{\bar{u}_2 + \int^t u_1\,dt\,\frac{\partial u_1}{\partial x} + \int^t w_1\,dt\,\frac{\partial u_1}{\partial z}}, \tag{5.1}$$

to the second order in the amplitude parameter. Restricting our attention to

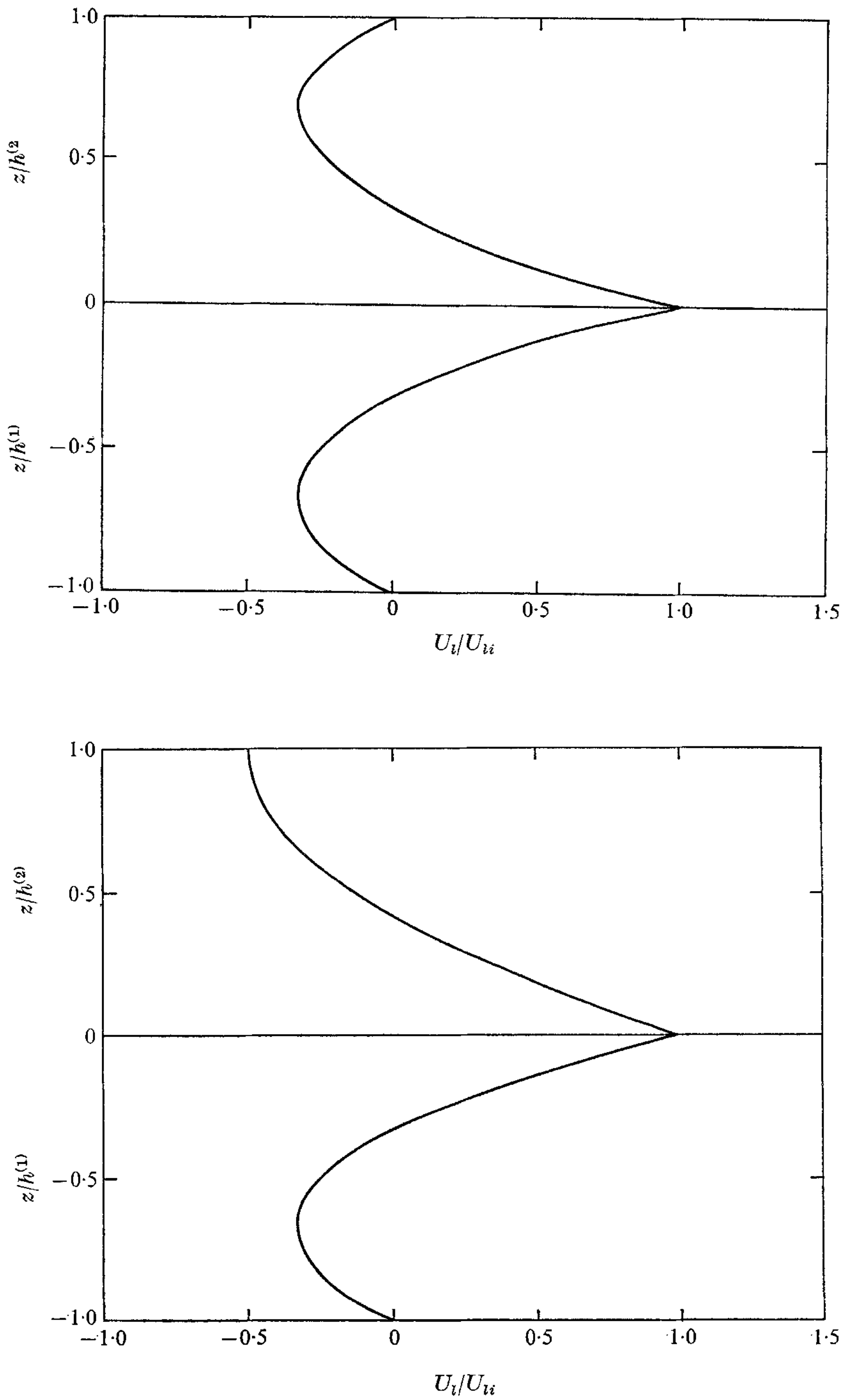


FIGURE 1. (*a*) Profile of horizontal mass transport velocity in a progressive wave for a two-layer fluid system bounded by two rigid horizontal planes. (*b*) Profile of horizontal mass transport velocity in a progressive wave for a two-layer fluid system whose uppermost surface is free.

the contribution to $U_l$ of order $\alpha^2\epsilon^{-1}$, we therefore have the following features of the induced flow.

### (*a*) *Fluid bounded by horizontal planes*

Within the lower boundary layer, adjacent to $z = -h^{(1)}$, $U_l$ is zero. In the interior of the lower fluid $U_l$ has the parabolic distribution of (4.12) with a stationary value (minimum) $U_l = -\frac{1}{3}U_{li}$ at $z = -\frac{2}{3}h^{(1)}$, a zero at $z = -\frac{1}{3}h^{(1)}$ and a value $U_l = U_{li} \geqslant 0$ at the edge of the interfacial layer where the transport velocity is greatest and in the direction of wave propagation for all possible values of the physical quantities involved. Within the layers, the horizontal mass transport velocity is given by

$$U_l = \alpha^2\epsilon^{-1}(\overline{\psi}'_{20} + \tfrac{1}{2}\psi_{10}\psi_{11}^{*''})\exp\{i(k-k^*)x\} = \alpha^2\epsilon^{-1}D\exp\{i(k-k^*)x\} \quad (5.2)$$

and so is constant through both layers (and continuous at the interface itself). The profile of the mass transport velocity in the upper fluid is similar in form to that in the lower fluid, becoming zero at the edge of the uppermost viscous layer, throughout which it is zero. The quantity $U_l/U_{li}$ is plotted against the vertical co-ordinate in figure 1(*a*).

### (*b*) *Fluid with free upper surface*

The profile of the mass transport velocity in the lower fluid is similar to that in case (*a*), though the interfacial velocity $U_{li}$ is different (but still positive). In the interior of the upper fluid $U_l$ has the parabolic distribution of (4.20) with a zero at $z/h^{(2)} = 1-\sqrt{(3)}/3$ and zero slope at the edge of the viscous surface layer, where $U_l = -\frac{1}{2}U_{li}$. The mass transport velocity is constant through the free surface layer taking the negative value $-\frac{1}{2}U_{li} = \alpha^2\epsilon^{(2)-1}\beta_{20}$, which thus determines the constant $\beta_{20}$ of (3.22). The form of $U_l/U_{li}$ as a function of the vertical co-ordinate is shown in figure 1(*b*).

In both cases, (*a*) and (*b*), the total horizontal flow $O(\alpha^2\epsilon^{-1})$ due to the mass transport velocity is zero in each fluid. Other parabolic velocity distributions containing some arbitrary constants but lacking the factor $\exp\{i(k-k^*)x\}$ can be added to those obtained above. However, these additional contributions to the mass transport do not decay as $x\to\infty$ where the waves themselves have vanished and we do not consider them further.

When the uppermost fluid surface is adjacent to a fixed rigid boundary, standard results of linear inviscid theory give

$$\alpha\Psi_{10}^{(1)} = ak_0\frac{\sinh k_0(z+h^{(1)})}{\sinh k_0 h^{(1)}}, \quad \alpha\Psi_{10}^{(2)} = -ak_0\frac{\sinh k_0(z-h^{(2)})}{\sinh k_0 h^{(2)}}, \quad (5.3)$$

$$\alpha\zeta_{10} = ak_0, \quad s = \frac{\sigma^2}{gk_0} = \frac{\rho^{(1)}-\rho^{(2)}}{\rho^{(1)}\omega^{(1)}+\rho^{(2)}\omega^{(2)}}, \quad \omega^{(r)} = \coth k_0 h^{(r)}, \quad (5.4)$$

so that

$$\alpha\Delta = ak_0(\omega^{(1)}+\omega^{(2)}). \quad (5.5)$$

Similarly, if the uppermost surface is free,

$$\alpha\Psi_{10}^{(1)} = a^{(1)}k_0\frac{\sinh k_0(z+h^{(1)})}{\sinh k_0 h^{(1)}}, \quad \alpha\Psi_{10}^{(2)} = a^{(1)}k_0\left[\frac{a_0^{(2)}}{a^{(1)}}\frac{\sinh k_0 z}{\sinh k_0 h^{(2)}} - \frac{\sinh k_0(z-h^{(2)})}{\sinh k_0 h^{(2)}}\right], \quad (5.6)$$

$$(\rho^{(1)}-\rho^{(2)})-s\rho^{(1)}(\omega^{(1)}+\omega^{(2)})+s^2(\rho^{(2)}+\rho^{(1)}\omega^{(1)}\omega^{(2)}) = 0, \tag{5.7}$$

$$a_0^{(2)}/a^{(1)} = s/(s\cosh k_0 h^{(2)}-\sinh k_0 h^{(2)}), \tag{5.8}$$

whence

$$\alpha\Delta = a^{(1)}k_0(\omega^{(1)}+\omega^{(2)})-a_0^{(2)}k_0 \operatorname{cosech} k_0 h^{(2)}. \tag{5.9}$$

Here, $a^{(1)}$ is the wave amplitude at the interface and $a_0^{(2)}$ that at the free surface according to linear inviscid theory. The lowest order mass transport velocity can now be calculated from § 4 in either of the above two cases.

## 6. Alternative formulation

The expansion scheme given in the previous sections provides a systematic procedure for obtaining higher order, steady, Eulerian velocity components $O(\alpha^2)$, $O(\alpha^2\epsilon)$, .... However, the scheme has some disadvantage in not demonstrating very clearly the physical processes by which the principal contribution $O(\alpha^2\epsilon^{-1})$ of the horizontal steady velocity arises. It therefore seems worthwhile to present here the following alternative and less formal account.

The essential feature of the considered fluid motion is that of a periodic interfacial wave decaying slowly with $x$ with logarithmic decrement $O(\epsilon)$. The main part of the oscillatory motion is irrotational except for interfacial (and top and bottom) boundary layers. From equation (2.3) and the expansion (3.1), the mean vorticity equation to second order in amplitude parameter $\alpha$ is

$$\overline{\left(\frac{\partial\psi_1}{\partial z}\frac{\partial}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial}{\partial z}\right)\nabla^2\psi_1} = \nu\nabla^4\overline{\psi}_2. \tag{6.1}$$

This will be an accurate approximation if the condition (3.14) is satisfied. Outside the boundary layers $\nabla^2\psi_1$ vanishes, whence

$$\nabla^4\overline{\psi}_2 = 0, \tag{6.2}$$

and the induced Eulerian mean velocity satisfies Stokes' equation in the main part of the fluid.

The $x$ component of the Navier–Stokes equation yields

$$\frac{\partial}{\partial x}(\overline{\rho u_1 u_1})+\frac{\partial}{\partial z}(\overline{\rho u_1 w_1})+\frac{\partial\overline{p}_2}{\partial x} = \mu\left(\frac{\partial^2\overline{u}_2}{\partial x^2}+\frac{\partial^2\overline{u}_2}{\partial z^2}\right). \tag{6.3}$$

Within one of the interfacial boundary layers the first term on the left is $O(\alpha^2\epsilon)$, whilst the second is $O(\alpha^2\epsilon^{-1})$ since $u_1$ varies rapidly through the layer. Similarly, use of the corresponding $z$ component equation and matching with the outer flow shows that the pressure term in (6.3) is $O(\alpha^2\epsilon)$. Thus, on integration,

$$\mu\frac{\partial\overline{u}_2}{\partial z}-\overline{\rho u_1 w_1} = \overline{\tau}_{zx} = \text{constant}+O(\alpha^2\epsilon^2), \tag{6.4}$$

where $\overline{\tau}_{zx}$ is the principal contribution to a total stress component in the layer and the decay factor is suppressed. Equation (3.18) shows that

$$\left(\mu\frac{\partial\overline{u}_2}{\partial z}+\overline{\zeta_1\mu\frac{\partial^2 u_1}{\partial z^2}}\right)$$

is continuous on the equilibrium level $z = 0$ of the interface. But

$$\overline{\zeta_1 \mu \frac{\partial^2 u_1}{\partial z^2}} = \overline{\zeta_1\left(\rho \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial x}\right)} + O(\alpha^2\epsilon^2) = -\rho\overline{u_1 w_1} + \overline{\zeta_1 \frac{\partial p_1}{\partial x}} + O(\alpha^2\epsilon^2).$$

Thus

$$\left(\bar{\tau}_{zx} + \overline{\zeta_1 \frac{\partial p_1}{\partial x}}\right) \tag{6.5}$$

is continuous on $z = 0$. Now according to the linear theory, the normal stress condition at the interface requires

$$p_1^{(1)} - p_1^{(2)} = (\rho^{(1)} - \rho^{(2)}) g \zeta_1 \quad \text{on} \quad z = 0, \tag{6.6}$$

surface tension being neglected, so that equation (6.5) gives

$$\bar{\tau}_{zx}^{(1)} - \bar{\tau}_{zx}^{(2)} = \tfrac{1}{2} g \kappa \zeta_1 \zeta_1^* (\rho^{(1)} - \rho^{(2)}) \quad \text{on} \quad z = 0, \tag{6.7}$$

where $\kappa = \mathrm{Im}(k)$ is the logarithmic decrement associated with spatial decay of the wave motion and is $O(\epsilon)$. [That the stress $\bar{\tau}_{zx}$ is discontinuous across $z = 0$ should come as no surprise—so also is the Eulerian mean velocity $\bar{u}_2$ according to equation (3.17). Both $\bar{\tau}_{zx}$ and $\bar{u}_2$ are, of course, continuous across the actual position $z = \zeta$ of the interface.] Equations (6.4) and (6.7) now show that the change in $\bar{\tau}_{zx}$ across the whole interfacial boundary-layer region is given by

$$[\bar{\tau}_{zx}] = \left[\mu \frac{\partial \bar{u}_2}{\partial z} - \rho \overline{u_1 w_1}\right] = \tfrac{1}{2} g \kappa \zeta_1 \zeta_1^* (\rho^{(1)} - \rho^{(2)}), \tag{6.8}$$

the term on the right being $O(\alpha^2 \epsilon)$.

To obtain the corresponding change in the viscous stress term $\mu \partial \bar{u}_2 / \partial z$ we proceed as follows. Denoting outer flow quantities by capital letters we have

$$\rho \frac{\partial U_1}{\partial t} + \frac{\partial P_1}{\partial x} = O(\alpha\epsilon^2). \tag{6.9}$$

Let $\zeta_1$ now be associated with the vertical displacement of a fluid particle. Then

$$\rho \overline{\zeta_1 \frac{\partial U_1}{\partial t}} + \overline{\zeta_1 \frac{\partial P_1}{\partial x}} = -\rho \overline{\frac{\partial \zeta_1}{\partial t} U_1} + \overline{\zeta_1 \frac{\partial P_1}{\partial x}} = O(\alpha^2\epsilon^2), \tag{6.10}$$

so that

$$\rho \overline{U_1 W_1} = \overline{\zeta_1 \frac{\partial P_1}{\partial x}} + O(\alpha^2\epsilon^2). \tag{6.11}$$

Combined use of equations (6.5), (6.8) and (6.11) now shows that

$$\left[\mu \frac{\partial \bar{u}_2}{\partial z}\right] = -\int \frac{\partial}{\partial z}\left(\overline{\zeta_1 \frac{\partial p_1}{\partial x}}\right) dz + O(\alpha^2\epsilon^2), \tag{6.12}$$

where the integral is to be taken (in two parts) across the whole interfacial boundary-layer region. It can readily be shown that

$$-\int \frac{\partial}{\partial z}\left(\overline{\zeta_1 \frac{\partial p_1}{\partial x}}\right) dz = -c_0^{-1} \int \overline{u_1 \frac{\partial p_1}{\partial x}}\, dz + O(\alpha^2\epsilon^2), \tag{6.13}$$

where $c_0$ is the (inviscid) phase velocity $\sigma/k_0$. But the main contribution to the dissipation integral *across the interfacial layers* is

$$\int \overline{\mu\left(\frac{\partial u_1}{\partial z}\right)^2} dz = -\int \overline{u_1 \frac{\partial p_1}{\partial x}} dz + O(\alpha^2\epsilon^2) \tag{6.14}$$

after an integration by parts, since $u_1$ and $\mu\, \partial u_1/\partial z$ are continuous on $z = 0$.

Thus,
$$\left[\mu \frac{\partial \overline{u}_2}{\partial z}\right] = c_0^{-1} \int \overline{\mu\left(\frac{\partial u_1}{\partial z}\right)^2} dz + O(\alpha^2\epsilon^2) \tag{6.15}$$

and calculation of the integral on the right-hand side confirms the value of $[\mu\partial\overline{u}_2/\partial z]$ as obtained from the formal analysis of §4.

Integrating again across a single interfacial layer we have from equation (6.4)

$$\mu\overline{u}_2 - \rho \int^z \overline{u_1 w_1} dz = \text{const.} + \overline{\tau}_{zx} z + O(\alpha^2\epsilon^3). \tag{6.16}$$

But
$$\rho \int^z \overline{u_1 w_1} dz = \rho \int^z \overline{u_1 \frac{\partial \zeta_1}{\partial t}} dz = -\rho \int^z \overline{\zeta_1 \frac{\partial \overline{u}_1}{\partial t}} dz$$
$$= -\mu \overline{\zeta_1 \frac{\partial u_1}{\partial z}} + O(\alpha^2\epsilon^2)$$

so that, in consequence of equation (6.16),

$$\overline{u}_2 + \overline{\zeta_1 \frac{\partial u_1}{\partial z}} = \text{const.} + O(\alpha^2). \tag{6.17}$$

Since the expression on the left is continuous on $z = 0$ and since $\partial u_1/\partial z$ is $O(\alpha)$ at the outer edge of the interfacial layer, we have

$$[\overline{u}_2] = O(\alpha^2) \tag{6.18}$$

across the whole interfacial region. This conclusion is in accordance with the results of §4.

In consequence of (6.15), $\partial\overline{u}_2/\partial z$ is different and $O(\alpha^2\epsilon^{-1})$ at the two outer edges of the interfacial layers. This property, together with (6.2), (6.18) and the boundary conditions on $\overline{u}_2$ at the top and bottom, establishes the existence of an induced Eulerian mean velocity of order $\alpha^2\epsilon^{-1}$. The essential physical feature is the change across the whole interfacial boundary layer region in the viscous stress $\mu\partial\overline{u}_2/\partial z$.

## 7. Discussion

In general, the calculated term $O(\alpha^2\epsilon^{-1})$ will be the principal contribution to the horizontal Lagrangian drift velocity in the fluid system. When

$$\eta = \frac{\rho^{(1)} - \rho^{(2)}}{\rho^{(1)}} \ll 1, \tag{7.1}$$

and the uppermost surface is free, the present calculation yields the main contribution to the mass transport velocity associated with the internal wave. But if $kh^{(r)} = O(1)$ on the scale of $\epsilon$ and $\eta$, we have $\sigma^2 h/g = O(\eta)$ for this wave and require

$$\eta \gg \frac{\nu^2}{gh^3} \tag{7.2}$$

to fulfil the condition $\epsilon \ll 1$. However, in order that the interfacial transport velocity in the internal wave be of more importance than that in the surface wave (with the same wavelength), it will be necessary that

$$\eta^{\frac{3}{4}} \gg \left(\frac{\nu^2}{gh^3}\right)^{\frac{1}{4}} \frac{a_s^2}{a_i^2} \tag{7.3}$$

where the subscripts $s$, $i$ denote interfacial amplitudes in the surface and internal waves respectively. The condition (7.3) can be satisfied for a considerable range of the physical quantities involved.

The analysis of previous sections has yielded the distribution of the lowest order mass transport velocity in interfacial conditions. It can, however, also be adapted to study the case of standing waves (considered as the combination of two progressive waves attenuated in opposite directions). A further application could be carried out in the case of multiple-layered fluid systems. There are no outstanding difficulties, other than algebraic, so that we shall not pursue the topic here.

The restriction on wave amplitude ($\alpha \ll \epsilon$) is, of course, a severe one, so that the present work can only be regarded as a preliminary analysis. In practice, it is possible that the range of validity of the present conduction solution may be rather better than is implied by equation (3.14), as has been indicated by Longuet-Higgins (1953) and the experiments of Russell & Osorio (1957) for progressive waves in a single fluid layer. The case when $\alpha \gg \epsilon$ was considered by Longuet-Higgins (1953). However, he made no allowance for the presence of double boundary layers, the likely necessity for which has been shown by Riley (1965) and Stuart (1966), at least in the neighbourhood of rigid boundaries. A different, possible application of the present theory is to particle velocities in internal waves in oceanographical conditions. When there is a single abrupt thermocline it is known that the first internal mode can be represented to good approximation on the basis of inviscid theory by the internal mode of an appropriate two-layer fluid system. The present results are related to a viscous phenomenon but may be of some interest, at least qualitatively, in the thermocline region provided the density gradients there are sufficiently large. This last point deserves further investigation.

According to inviscid theory, the Lagrangian drift velocity would be $O(\alpha^2)$. The term $O(\alpha^2\epsilon^{-1})$ found in the present work is set up by diffusion of vorticity outwards from the interfacial layers. In this respect, it is of interest to note the following point concerning the induced flow in the bottom boundary layer. The drift velocity there is $O(\alpha^2)$ and is influenced by the $O(\alpha^2\epsilon^{-1})$ outer flow to the extent that a linear term in the magnified variable $(z+h^{(1)})/\epsilon^{(1)}$ is required, since the $O(\alpha^2\epsilon^{-1})$ vorticity at the outer edge of this layer is non-zero by equations (4.12) and (4.20). The possible presence of this term has been discussed in another context by Riley (1965) and Stuart (1966), but in both of these works, such a term was not required.

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